MILP FOR CMDTSP

The CMDTSP can be formulated as a Mixture Integer Linear Programming problem. Suppose there are m depots, n cities, l stations, and r_s units resources for station s. We use E to denote each salesman's initial energy level and maximum energy level. For simplicity, we use $[x]$ to denote the set $\{z \in \mathbb{N}_+, z \leq x\}$ in this section.

Let $\beta_{a,u,v,i,j} \in \{0,1\}$, be a binary variable indicating whether a salesman who starts from depot a passes the edge between node i and node j when traveling from depot or city u to depot or city v. Then the objective function is to minimize the total weights of selected edges:

$$\min \sum_{a=1}^{m} \sum_{u,v=1}^{m+n} \sum_{i,j=1}^{m+n+l} c_{i,j} \cdot \beta_{a,u,v,i,j}. \tag{4}$$

Let $\gamma_{a,u} \in \{0,1\}$ be the binary variable indicating whether depot or city u is visited by the salesman who starts from depot a. Then each city should be visited exactly once, which can be expressed as constraints:

$$\sum_{a=1}^{m} \gamma_{a,u} = 1, \qquad u \in [m+n], \tag{5a}$$

$$\gamma_{a,a} = 1, \qquad a \in [m]. \tag{5b}$$

To guarantee the TSP properties of salesmen, we adapt the GG formulation [?] with binary $\phi_{a,u,v} \in \{0,1\}$, which indicates whether the tour starts from depot a will consecutively visit depots or cities u and v, and real variable $f_{a,u,v} \in \mathbb{R}_+$ w.r.t. $\phi_{a,u,v}$. Eq. (6a)-(6b) are the constraints to handle different lengths of TSP tours given the city partition, while the rest constraints are standard GG formulations for each salesman.

$$\sum_{u=1}^{m+n} \phi_{a,u,v} = \gamma_{a,v}, a \in [m], v \in [m+n] \setminus \{a\}, \tag{6a}$$

$$\sum_{v=1}^{m+n} \phi_{a,u,v} = \gamma_{a,u}, a \in [m], u \in [m+n] \setminus \{a\}, \tag{6b}$$

$$\sum_{u=1}^{m+n} \phi_{a,u,a} = \sum_{v=1}^{m+n} \phi_{a,a,v}, a \in [m], \tag{6c}$$

$$\sum_{u=1}^{m+n} \phi_{a,u,a} \geq \gamma_{a,v}, a \in [m], v \in [m+n] \setminus \{a\}, \tag{6d}$$

$$\sum_{u=1}^{m+n} \phi_{a,u,a} \leq 1, a \in [m], \tag{6e}$$

$$\sum_{u=1}^{m+n} \gamma_{a,u} - \phi_{a,u,a} \geq 1, a \in [m], \tag{6f}$$

$$\phi_{a,u,u} = 0, a \in [m], u \in [m+n], \tag{6g}$$

$$\phi_{a,u,v} = 0, a \in [m], u \in [m+n], v \in [m], \tag{6h}$$

$$f_{a,u,v} \leq (m+n) \cdot \phi_{a,u,v}, a \in [m], u, v \in [m+n], \tag{6i}$$

$$\sum_{v=1}^{m+n} \gamma_{a,v} - f_{a,a,v} = 1, a \in [m], \tag{6j}$$

$$\sum_{u=1}^{m+n} f_{a,u,v} - f_{a,v,u} = \gamma_{a,v}, a \in [m], v \in [m+n] \setminus [m], \tag{6k}$$

$$f_{a,u,u} = 0, a \in [m], u \in [m+n], \tag{6l}$$

$$f_{a,u,v} = 0, a \in [m], u \in [m+n], v \in [m]. \tag{6m}$$

Given the city partition and the TSP property guarantee, we then add the following constraints to select edges in the whole graph to meet with them and get rid of some corner cases, such as self-loops, connected depots, etc. We define binary variables $\beta_{a,u,v,i,j} \in \{0,1\}$ for $a \in [m], u, v \in [m+n], i, \in [m+n+l]$ representing the selection of edge (i,j) during the visit between cities or depots u, v for salesman a.

The first constraint we need to meet is the resource limitation for each station, which is

$$\sum_{a=1}^{m} \sum_{u,v=1}^{m+n} \sum_{j=1}^{m+n+l} \beta_{a,u,v,m+n+s,j} \leq r_s, s \in [l]. \tag{7}$$

Then we should make the selection of edges consistent with previous TSP results:

$$\sum_{j=1}^{m+n+l} \beta_{a,u,v,u,j} = \phi_{a,u,v}, a \in [m], u, v \in [m+n], \tag{8a}$$

$$\sum_{i=1}^{m+n+l} \beta_{a,u,v,i,v} = \phi_{a,u,v}, a \in [m], u, v \in [m+n]. \tag{8b}$$

Next, we ask the edges selected between u and v to form a path, which means the in-degree and out-degree are at most 1 for each node and exactly 0 for the start and end nodes

$$\sum_{j=1}^{m+n+l} \beta_{a,u,v,i,j} \leq 1, a \in [m], u, v \in [m+n], \quad i \in [m+n+l], \tag{9a}$$

$$\sum_{i=1}^{m+n+l} \beta_{a,u,v,i,j} \leq 1, a \in [m], u, v \in [m+n], \quad j \in [m+n+l], \tag{9b}$$

$$\sum_{i=1}^{m+n+l} \beta_{a,u,v,i,u} = 0, a \in [m], u, v \in [m+n], \tag{9c}$$

$$\sum_{j=1}^{m+n+l} \beta_{a,u,v,v,j} = 0, a \in [m], u, v \in [m+n], \tag{9d}$$

and the in-degree should be equal to out-degree

$$\sum_{j=1}^{m+n+l} \beta_{a,u,v,i,j} - \beta_{a,u,v,j,i} = 0, \quad a \in [m], u, v \in [m+n], i \in [m+n+l] \setminus \{u, v\}. \tag{10}$$

Also, we need to get rid of self-loop

$$\beta_{a,u,v,i,i} = 0, a \in [m], u, v \in [m+n], \quad i \in [m+n+l], \tag{11}$$

Finally, we do not want to visit any other city or depots along the path from u to v, i.e.

$$\sum_{j=1}^{m+n+l} \beta_{a,u,v,i,j} = 0, a \in [m], u, v \in [m+n], \quad i \in [m+n+l] \setminus \{u, v\}. \tag{12}$$

To address the energy constraints, we define non-negative real variables e_u, representing the energy level of a salesman when visiting the depot or city u. Initially, the energy for all salesman is E,

$$e_a = E, a \in [m]. \tag{13}$$

Every salesman should not run out of energy. Basically,

$$e_u \geq 0, u \in [m+n] \tag{14}$$

In case of the edge (i, j) selected, if both i and j are stations, then the distance should not exceed the maximum distance a full energy salesman can travel,

$$c_{i,j} \cdot \beta_{a,u,v,i,j} \leq E, a \in [m], u, v \in [m+n], \\ i, j \in [m+n+l] \setminus [m+n]. \tag{15}$$

If the start node i is not a station but a city or depot u, then the distance between u j should not take more than the energy remaining at u to go,

$$c_{u,j} \cdot \beta_{a,u,v,u,j} \leq e[u], a \in [m], u, v \in [m+n], \\ j \in [m+n+l]. \tag{16}$$

And if the arriving node j is not a station but a city v, then the energy remains is bounded by full minus the consumption, which is

$$c_{i,v} \cdot \beta_{a,u,v,i,v} \leq E - e[v], a \in [m], u \in [m+n], \\ v \in [m+n] \setminus \{a\}, i \in [m+n+l] \setminus [m+n]. \tag{17}$$

If both i and j are not stations, i.e. $i = u$ and $j = v$, then

$$(c_{u,v} + E) \cdot \beta_{a,u,v,u,v} \leq E + e[u] - e[v], a \in [m], \\ u \in [m+n], v \in [m+n] \setminus \{a\}. \tag{18}$$

For the path returning back to the depot, where the arriving node has $u[v] = 0$, we can get two restrictions following the discussion above,

$$(c_{u,a} + E) \cdot \beta_{a,u,a,u,a} \leq E + e[u], a \in [m], u \in [m+n], \tag{19a}$$
$$c_{i,a} \cdot \beta_{a,u,a,i,a} \leq E, a \in [m], u \in [m+n], i \in [m+n+l]. \tag{19b}$$

Finally, combine objects (4) and constraints (5)-(19), we get the MILP formulation.